

The iterated Prisoner's Dilemma in societies of deterministic players

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Abstract

Two players engaged in the Prisoner's Dilemma have to choose between cooperation and defection, the pay-off of the players is determined by a weight $w = (T, R, P, S)$. For deterministic strategies p_1, \dots, p_n we consider a society $S = S(u_i : p_i \mid i = 1, \dots, n)$ formed by $u = \sum_{i=1}^n u_i$ individuals playing at random the IPD with weight w . We introduce the concept of a w -successful society as one where all individuals have eventually a non-negative pay-off. We discuss success of individuals and societies by means of quadratic forms associated to the pay-off matrix of the given set of strategies.

Keywords

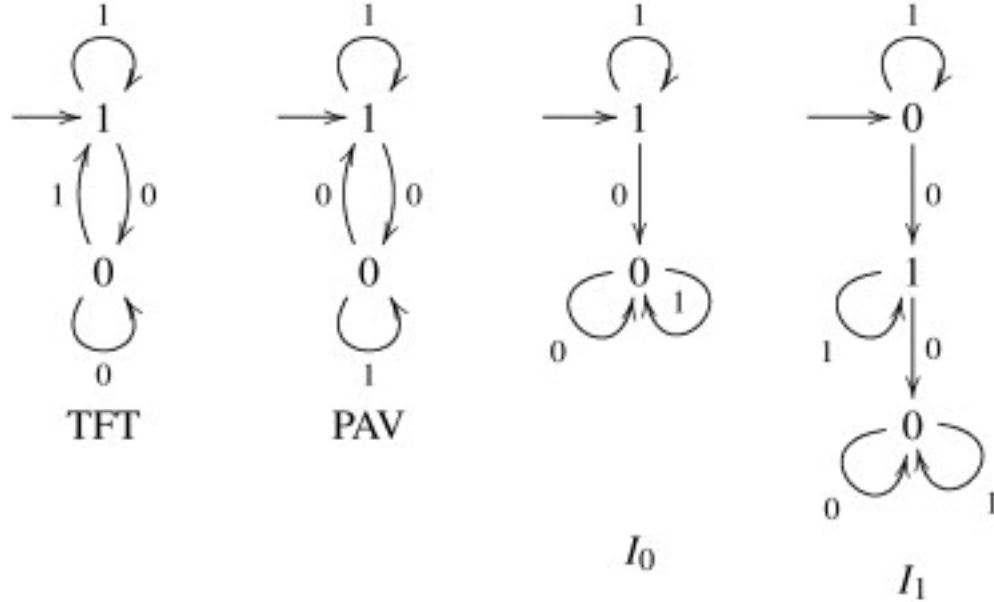
- Prisoner's Dilemma;
- Iterated Prisoner's Dilemma;
- Successful societies;
- Weakly positive quadratic forms

1. Introduction

Since its formulation in 1950, the Prisoner's Dilemma has become the leading metaphor to investigate rationales for cooperation (see [1], [4] and [7] for extensive literature lists). Two players engaged in the Prisoner's Dilemma have to choose between cooperation and defection. The players confront each other indefinitely often, receiving in each round R points if they both cooperate and P points if they both defect; moreover, a defector exploiting a cooperator receives T points, while the cooperator receives only S points. It is assumed that $T > R > 0 > P > S$ and $0 > T + S$, the last condition implying that it is not worth for a player to cooperate and defect alternatively while the coplayer is cooperating. A tuple $w = (T, R, P, S)$ satisfying the above conditions is called an *admissible weight*.

The iterated Prisoner's Dilemma (IPD) offers rich possibilities for ingenious strategies. Most of the literature on the topic deals only with stochastic strategies (see for example [7] and [8]). A *deterministic strategy* $p = (\{a_0, a_1, \dots, a_n\}, f_0, f_1, s)$ is given by a finite set $\{a_0, a_1, \dots, a_n\}$ of states, where a_0 is a distinguished *initial state*; f_0 and f_1 are *transition functions* of the states and s is the *outcome function* assigning 0 or 1 to each state, where 1 stands for cooperating and 0 for defecting. Hence a deterministic strategy is a finite automaton (see [2]).

Deterministic strategies may be depicted as finite oriented valued digraphs, as in the following examples, where \rightarrow indicates the initial state and the values of s are written on the vertices. Strategy TFT is the famous tit-for-tat strategy: cooperate in the first round, then do whatever the other did last time. Since the well-known Axelrod's tournaments [1], tit-for-tat has been considered the major paradigm of altruistic behaviour [4] and [5]. Strategy PAV (for Pavlov) was introduced by Nowak and Sigmund [9] and shown to outperform TFT in computer runned simulations of heterogeneous sets of probabilistic strategies. Our computer programs show that the *intolerant strategy* I_0 outperforms all deterministic strategies with two states and I_1 outperforms all deterministic strategies with at most three states.



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Let p_1, \dots, p_n be deterministic strategies and consider a *society* $S = S(u_i: p_i | i = 1, \dots, n)$ formed by $u = \sum_{i=1}^n u_i$ individuals playing at random the IPD with admissible weight w (i.e. in each round, two individuals are chosen randomly to play the next step of the corresponding IPD, each individual recalling their last play against one another and responding accordingly), among them, $0 < u_i$ individuals use strategy p_i . We shall assume that there is an unlimited number of rounds, all occurring with probability one. (For certain considerations of the IPD it is assumed, see [1], that the next round happens with probability $w < 1$. The limiting case $w = 1$ is usually of great interest, see [8] for a discussion). Many interesting problems arise from the consideration of the *terminal pay-off* $g_S(x) = \lim_{t \rightarrow \infty} \frac{g_t(x)}{t}$ of an individual x in the society S , where $g_t(x)$ is the pay-off accumulated by x in the first t rounds. Observe that, for the sake of simplicity, we omit the dependence on the parameter w , but we may write $g_S^{(w)}(x)$ for $g_S(x)$ if we want to stress the parameter w of the IPD.

In Section 2, we show that in case the individual x uses strategy p_i , then

$$g_S(x) = \frac{2}{u(u+1)} \left[(u_i - 1)g(p_i : p_i) + \sum_{i \neq j} u_j g(p_i : p_j) \right],$$

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where $g(p_i : p_j)$ is the *terminal pay-off* of p_i relative to p_j . We shall consider the *pay-off matrix* of the society S as the $n \times n$ -matrix $G = (g(p_i : p_j))$ (we write $G^{(w)} = (g^{(w)}(p_i : p_j))$ in case we want to make explicit the parameter w). According to Maynard Smith [6], a strategy is *evolutionary stable* if an infinite homogeneous population adopting it (i.e. $n = 1$ and $S = S(u_1 : p_1)$, for $u_1 \gg 0$) cannot be invaded by mutants. We generalize the concept and say that a society $S = S(u_i : p_i | i = 1, \dots, n)$ is *stable* if any individual of S performs, at the long run, better keeping its strategy than changing to a new one. In Section 3 we prove that for a set p_1, \dots, p_n of retaliatory strategies with $g(p_i : p_j) \geq 0$ ($1 \leq i, j \leq n$), any society $S(u_i : p_i | i = 1, \dots, n)$ is stable. We recall that a strategy $p = (\{a_0, a_1, \dots, a_n\}, f_0, f_1, s)$ is *retaliatory* if $s(f_0(a)) = 0$ for any state a . This result generalizes the observation in [1] that TFT is an ESS. We shall say that a society $S = S(u_i : p_i | i = 1, \dots, n)$ is *w-successful* if for any individual x of S we have $g_S^{(w)}(x) \geq 0$. It will be easy to show that S is *w-successful* if and only if $G^{(w)}u \geq (g^{(w)}(p_i : p_i) : i = 1, \dots, n) = g^{(w)}$ as column vectors. This gives conditions on the matrix $G^{(w)}$ for the existence of vectors u with all entries $u_i > 0$, ($i = 1, \dots, n$) such that $S(u_i : p_i | i = 1, \dots, n)$ is *w-successful*. We shall say that S is *w-macro-successful* if $\sum_{x \in S} g_S^{(w)}(x) = \sum_{i=1}^n u_i g_S^{(w)}(x_i) \geq 0$, for any selection of individuals x_i with strategy p_i ($1 \leq i \leq n$). Clearly, an individual x is *w-successful* in the society S if at the long run its pay-off increases. In the same way, the society S is *w-macro-successful* if the total pay-off (the sum of the pay-offs of its 'citizens') eventually increases. In Section 4, we introduce the quadratic form $q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$ associated with the symmetric matrix $\frac{1}{2}(G^{(w)} + (G^{(w)})^t)$ and show that $S(u_i : p_i | i = 1, \dots, n)$ is *w-macro-successful* if and only if $q_{(p_1, \dots, p_n)}^{(w)}(u_1, \dots, u_n) \geq (u_1, \dots, u_n)^t g^{(w)}$. Finally, $S(u_i : p_i | i = 1, \dots, n)$ is *w-macro-successful* for any choice of numbers u_1, \dots, u_n with big enough $u = \sum_{i=1}^n u_i$ if and only if the quadratic form $q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$ is *weakly positive*, that is $q_{(p_1, \dots, p_n)}^{(w)}(v_1, \dots, v_n) > 0$ for any vector $0 \neq (v_1, \dots, v_n) \in \mathbb{N}^n$. We give conditions on the matrix $\frac{1}{2}(G^{(w)} + (G^{(w)})^t)$ characterizing the weak positivity of $q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$. Clearly, the concepts of success for individuals in a society S or that of successful societies depend on the chosen parameters T, R, P and S . The relativity of the concepts stresses the fact

that the pay-off of strategies playing the IPD depend as much on the structure of the strategies themselves as on the setting of the game. Observe the particular role played by 0 in the definitions: an individual playing the strategy p against an individual playing the strategy p' is w -successful (in this game) if and only if the pay-off $g^{(w)}(p:p') \geq 0$. For further remarks see Section 5.

2. Deterministic strategies

2.1.

Recall that a *deterministic strategy* p is a tuple $(\{a_0, a_1, \dots, a_n\}, f_0, f_1, S)$ where $\{a_0, a_1, \dots, a_n\}$ is a finite set of states, with a_0 a distinguished *initial state*, $f_0 : \{a_0, a_1, \dots, a_n\} \rightarrow \{a_0, a_1, \dots, a_n\}$ and $f_1 : \{a_0, a_1, \dots, a_n\} \rightarrow \{a_0, a_1, \dots, a_n\}$ are *transition functions* of the states and $s : \{a_0, a_1, \dots, a_n\} \rightarrow \{0, 1\}$ the *outcome function*.

Given two deterministic strategies

$$p = (\{a_0, \dots, a_n\}, f_0, f_1, S)$$

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$$\text{and } p' = (\{b_0, \dots, b_m\}, f'_0, f'_1, t),$$

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define the *tournament* $t(p:p')$ as an oriented graph with j th vertex $x_j = a'_j : b'_j$, where a'_j is a state of p , b'_j is a state of p'

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with $a'_0 = a_0$, $b'_0 = b_0$ and arrows

$$x_j = a'_j : b'_j \xrightarrow{\alpha_j} x_{j+1} = f_{t(b'_j)}(a'_j) : f'_{s(a'_j)}(b'_j).$$

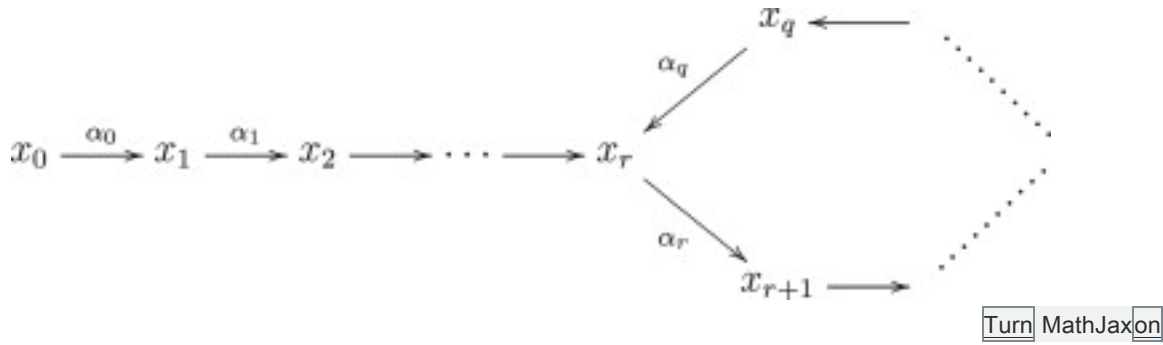
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We identify a_j with $s(a'_j) : t(b'_j)$. In other words, $t(p:p')$ is the orbit of (a_0, b_0) under the function

$$f \times f' : \{a_0, \dots, a_n\} \times \{b_0, \dots, b_m\} \rightarrow \{a_0, \dots, a_n\} \times \{b_0, \dots, b_m\}$$

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where $(f \times f')(a_i, b_j) = (f_{t(b_j)}(a_i), f'_{s(a_i)}(b_j))$. Therefore $t(p:p')$ has the shape



with $q \leq nm$. It is clear that

$$\lim_{t \rightarrow \infty} \frac{1}{t} g_{(t)}(p : p') = \frac{1}{c(p : p')} \sum_{i=r}^q g(\alpha_i) =: g(p : p') \text{ for } c(p : p') = q - r + 1$$

is the *terminal pay-off* of p relative to p' , where $g_{(t)}(p : p') = \sum_{i=0}^t g(\alpha_i)$ and $g_i(p : p') = g(\alpha_i)$ is the pay-off of p relative to p' at the i th step of the IPD (where $g(1:1) = R$, $g(0:0) = P$, $g(0:1) = T$ and $g(1:0) = S$) and $c(p : p')$ is the length of the cycle in the orbit. For examples see Section 5.

2.2.

Let S be a society with u individuals. Society S plays random IPD as follows: consider two different individuals x and y , x playing with strategy p and y playing with strategy p' . At round t , the couple (x, y) may not be confronted, then the pay-off $g_t(x : y) = 0$. In case x and y are confronted for the j th time, then the tournament $t(p : p')$ yields the arrow

$$a'_{j-1} : b'_{j-1} \xrightarrow{\alpha_{j-1}} a'_j : b'_j$$

and therefore $g_t(x : y) = g_j(p : p')$. That is, each player keeps track of past play against all individual players.

Lemma.

The expected value $g_t(x : y)$ is

$$g_t(x : y) = \frac{1}{u} \left(1 - \frac{1}{u}\right)^{t-1} \left[\sum_{k=1}^t \binom{t-1}{k-1} \left(\frac{1}{u-1}\right)^{k-1} g_k(p : p') \right]$$

where $\underline{u} = \frac{1}{2}u(u-1)$.

Proof.

Let $p(t, k)$ be the probability to select the couple (x, y) at the round t for the k th time. Out of \underline{u}^t possible selections of couples, couple (x, y) is selected k times, the other $t - k$ times any of the remaining $\underline{u} - 1$ couples is selected. Then

$$p(t, k) = \frac{1}{\underline{u}^t} \binom{t-1}{k-1} (\underline{u} - 1)^{t-k}.$$

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Hence,

$$g_t(x : y) = \sum_{k=1}^t p(t, k) g_k(p : p') = \frac{1}{\underline{u}^t} \sum_{k=1}^t \binom{t-1}{k-1} (\underline{u} - 1)^{t-k} g_k(p : p'),$$

which is the desired expression. \square

2.3.

There is an interesting consequence of (2.2): an individual x with strategy $(\{a_0, \dots, a_n\}, t_0, t_1, S)$ may profit for a long while from a confident homogeneous society S acting with strategy $p' = (\{b_0, \dots, b_m\}, g_0, g_1, S')$ if $s(a_0) = 0$, $s'(b_0) = 1$ and S is large enough. More precisely, let u be the number of individuals in S .

Lemma.

We have $g_{(t)}(x : y) \geq 0$ as long as $t \leq \ln\left(1 - \frac{T}{S}\right) [\ln \underline{u} - \ln(\underline{u} - 1)]^{-1} + 1$.

Proof.

By hypothesis $g_1(p : p') = T$ and clearly, $g_i(p : p') \geq S$ for $i \geq 2$. Then

$$\begin{aligned} g_{(t)}(x : y) &\geq \left(1 - \frac{1}{\underline{u}}\right)^t \left[\frac{T}{\underline{u} - 1} + S \sum_{k=2}^t \binom{t-1}{k-1} (\underline{u} - 1)^{-k} \right] \\ &= \left(1 - \frac{1}{\underline{u}}\right)^t \left\{ \frac{T}{\underline{u} - 1} + \frac{S}{\underline{u} - 1} \left[\left(\frac{\underline{u}}{\underline{u} - 1}\right)^{t-1} - 1 \right] \right\}. \end{aligned}$$

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Therefore $g_{(t)}(x : y) \geq 0$ if and only if $\left(\frac{\underline{u}}{\underline{u} - 1}\right)^{t-1} \leq 1 - \frac{T}{S}$. \square

For a numerical example, consider $T = 2$, $S = -3$, for $u \gg 0$, $\ln \underline{u} - \ln(\underline{u} - 1) \approx \frac{1}{\underline{u}}$, then $g_{(t)}(x : y) > 0$ for $t \leq 0.51 \underline{u}$.

2.4.

The next Proposition only expresses the fact that, after a preperiod, all the confrontations between individuals enter in a tournament-cycle determined by their strategies.

Proposition.

Let $S = S(u_i : p_i | i = 1, \dots, n)$ and x be an individual of S with strategy p_i . Let $\underline{u} = \sum_{j=1}^n u_j$ be the total population of S . Then

$$g_S(x) = \lim_{t \rightarrow \infty} \frac{g^{(t)}(x)}{t} = \frac{1}{\underline{u}} \left[(u_i - 1)g(p_i : p_i) + \sum_{j \neq i} u_j g(p_i : p_j) \right].$$

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Proof.

Let x_j be an individual of S with strategy p_j . Then

$$g_S(x) = (u_i - 1) \lim_{s \rightarrow \infty} \frac{g^{(s)}(x : x_i)}{s} + \sum_{j \neq i} u_j \lim_{s \rightarrow \infty} \frac{g^{(s)}(x : x_j)}{s}$$

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if all limits exist.

Recall that

$$g^{(s)}(x : x_j) = \sum_{t=1}^s g_t(x : x_j) \quad \text{and} \quad g_t(x : x_j) = \frac{1}{\underline{u}^t} \sum_{k=1}^t \binom{t-1}{k-1} (\underline{u} - 1)^{t-k} g_k(p_i : p_j).$$

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Moreover $g_{k_0+k_0+m}(p_i : p_j) = g_{k_0+m}(p_i : p_j)$, for c the length of the tournament-cycle, $c = c(p_i : p_j) > m \geq 0$ and k_0 the length of the preperiod in the tournament $t(p : p')$.

Consider $|g_k(p_i : p_j)| \leq \gamma$ for $k = 1, \dots, k_0$ and

$$c(t, k_0) = \sum_{k=1}^{k_0} \binom{t-1}{k-1} (\underline{u} - 1)^{t-k}, \quad d(t, k_0) = \sum_{k=k_0+1}^t \binom{t-1}{k-1} (\underline{u} - 1)^{t-k},$$

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$$\underline{c}(t, k_0) = \sum_{k=1}^{k_0} \binom{t-1}{k-1} \left(\frac{1}{\underline{u} - 1} \right)^{k-1}, \quad \underline{d}(t, k_0) = \sum_{k=k_0+1}^t \binom{t-1}{k-1} \left(\frac{1}{\underline{u} - 1} \right)^{k-1},$$

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hence $\underline{c}(t, k_0) = \left(\frac{1}{\underline{u} - 1} \right)^{t-1} c(t, k_0)$, $\underline{d}(t, k_0) = \left(\frac{1}{\underline{u} - 1} \right)^{t-1} d(t, k_0)$.

Therefore

$$\left| \frac{1}{s} g^{(s)}(x : x_j) \right| = \left| \frac{1}{s} \sum_{t=1}^s g_t(x : x_j) \right| \leq \frac{\gamma}{s} \sum_{t=1}^s \frac{1}{\underline{u}} \left(1 - \frac{1}{\underline{u}} \right)^{t-1} [\underline{c}(t, k_0) + \underline{d}(t, k_0)] = \frac{\gamma}{\underline{u}},$$

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and $\lim_{t \rightarrow \infty} \frac{g(t, k_0)}{d(t, k_0)} = \lim_{t \rightarrow \infty} \frac{c(t, k_0)}{d(t, k_0)} = 0$.

Hence

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{1}{s} g_{(s)}(x : x_j) &= \frac{1}{u} \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{t=1}^s \left(1 - \frac{1}{u}\right)^{t-1} \left(\sum_{k=k_0}^t \binom{t-1}{k-1} \left(\frac{1}{u-1}\right)^{k-1} g_k(p_i : p_j) \right) \\ &= \frac{1}{u} \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{t=1}^s \left(1 - \frac{1}{u}\right)^{t-1} \left[\sum_{\ell=0}^{c-1} \left\{ \sum_{k=k_0+r c+m} \binom{t-1}{k-1} \left(\frac{1}{u-1}\right)^{k-1} \right\} \right] \\ &= \frac{1}{u} \lim_{s \rightarrow \infty} \frac{1}{s} \sum_{t=1}^s \left(1 - \frac{1}{u}\right)^{t-1} \frac{1}{c} \left(1 + \frac{1}{u-1}\right)^{t-1} \left(\sum_{\ell=0}^{c-1} g_{k_0+\ell}(p_i : p_j) \right) \end{aligned}$$

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The proof is complete. \square

3. Stability

3.1.

Let $w = (T, R, P, S)$ be an admissible weight. In the next sections it will be of importance to make explicit the parameter w . Let p_1, \dots, p_n be deterministic strategies. A society $S = S(u_i : p_i | i = 1, \dots, n)$ is said to be w -stable if for every individual x in S using strategy p_i and any other strategy p_0 defining a society S'

$= S(I : p_0; u_1 : p_1; \dots; u_i - I : p_i; \dots; u_n : p_n)$ we have $g_{S'}^{(w)}(x') \leq g_S^{(w)}(x)$, for the individual x in S' with strategy p_0 . This translates to the condition: for every strategy p_0 , we have

$$(u_i - 1)g^{(w)}(p_0 : p_i) + \sum_{j \neq i} u_j g^{(w)}(p_0 : p_j) \leq (u_i - 1)g^{(w)}(p_i : p_i) + \sum_{j \neq i} u_j g^{(w)}(p_i : p_j)$$

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In the case of a homogeneous society ($n = 1$), this is equivalent to

$$g^{(w)}(p_0 : p_1) \leq g^{(w)}(p_1 : p_1)$$

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which is a condition in the limit of ESS as defined in [6].

3.2.

We recall that a strategy $p = (\{a_0, a_1, \dots, a_n\}, f_0, f_1, S)$ is *nice* if $s(a_0) = 1$ and $s(f_1(a_j)) = 1$, for $1 \leq j \leq n$; p is *retaliatory* if $s(f_0(a_j)) = 0$, for $1 \leq j \leq n$. We shall say that p is w -self-supportive if $g^{(w)}(p : p) > 0$.

Lemma.

Let p, p' be two strategies. Then

(i)

Always $g^{(w)}(p:p) \leq R$. If p is nice, then $g^{(w)}(p:p) = R$ and hence p is w -self-supportive.

(ii)

If p and p' are retaliatory, then either $g^{(w)}(p:p') = R$ or $g^{(w)}(p:p') \leq 0$.

(iii)

If p is retaliatory, then p is w -self-supportive if and only if $g^{(w)}(p:p) = R$.

Proof.

(i) The cycle in $t(p:p)$ has a arrows of the form $1:1$ and b of the form $0:0$.

Then $g^{(w)}(p:p) = \frac{aR+bP}{a+b} \leq \frac{aR}{a+b} \leq R$. If the strategy p is nice, when paired with itself, it will cooperate indefinitely, resulting in the average pay-off of R .

(ii) The cycle in $t(p:p')$ has a arrows of the form $1:1$, b of the form $0:0$, c of the

form $0:1$ and d of the form $1:0$. Then $g^{(w)}(p:p') = \frac{aR+bP+cT+dS}{a+b+c+d}$. If both p and p' are retaliatory, then $b > 0$ implies that $a=c=d=0$ and $g^{(w)}(p:p') < 0$. If $c > 0$,

then $c=d$ and $a=b=0$, implying that $g^{(w)}(p:p') = \frac{T+S}{2} < 0$. Indeed, in this case, the tournament-cycle has an arrow corresponding to the outcome $0:1$ which implies that $0:0$ is not a possible outcome, by the first considered case. Then after $0:1$, the second player retaliates and only $1:0$ is a possible outcome. This repeats over to show that there are the same number of $0:1$ outcomes as $1:0$ outcomes in the cycle, or $c=d$. Similarly, if $d > 0$, then $b=d$ and $a=c=0$ and $g^{(w)}(p:p') = \frac{P+S}{2} < 0$.

(iii) follows from (i) and (ii). \square

Proposition 3.3.

Let p_1, \dots, p_n be retaliatory strategies such that $g^{(w_0)}(p_i:p_j) \geq 0$, for any $1 \leq i, j \leq n$ and some admissible weight w_0 .

Then $S(u_i:p_i | i=1, \dots, n)$ is w -stable for any admissible weight w and any vector $u \in N^n$.

Proof.

By 3.2, for any admissible weight $w = (T, R, P, S)$, we have $g^{(w)}(p_i:p_j) = R$, for all $1 \leq i, j \leq n$, and for any other strategy p_0 , we have $g^{(w)}(p_0:p_i) \leq R$. Then

$$\begin{aligned} (u_1 - 1)g^{(w)}(p_0:p_1) + \sum_{j \neq 1} u_j g^{(w)}(p_0:p_j) &\leq \left[(u_1 - 1) + \sum_{j \neq 1} u_j \right] R \\ &= (u_1 - 1)g^{(w)}(p_1:p_1) + \sum_{j \neq 1} u_j g^{(w)}(p_1:p_j) \end{aligned}$$

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\square

4. Successful societies

4.1.

Let $w = (T, R, P, S)$ be an admissible weight. Let p_1, \dots, p_n be deterministic strategies and $G^{(w)} = (g^{(w)}(p_i, p_j))$ the *terminal pay-off* $n \times n$ -matrix.

Let $S = S(u_i, p_i | i = 1, \dots, n)$ be a society corresponding to the given strategies. Then an individual x in S is said to be *w-successful in the society S* if the terminal pay-off $g_S^{(w)}(x) \geq 0$. If x uses the strategy p_i , this is equivalent to

$$(G^{(w)}u)_i \geq g^{(w)}(p_i, p_i),$$

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where u is the column vector with i th entry u_i , that is, the individual x gets a higher pay-off from being part of the society S than if it were to form a society with individuals playing the same strategy p_i .

Proposition.

The society $S = S(u_i, p_i | i = 1, \dots, n)$ is *w-successful* if and only if

$$G^{(w)}u \geq g^{(w)}$$

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where $g^{(w)}$ is the column vector whose i th entry is $g^{(w)}(p_i, p_i)$. \square

4.2.

Consider the vector space $V = R^n$. A *cone* K in V is a closed subset satisfying: (i) $0 \in K$, (ii) for $v \in K$ and $\lambda \geq 0$, then $\lambda v \in K$, (iii) if $v, v' \in K$, then $v + v' \in K$. The cone K is said to be *proper* if $K \cap (-K) = \{0\}$ and is said to be *solid* if it contains a basis of V .

The set V^+ of vectors v with non-negative coordinates is a solid proper cone in V . Given a linear transformation $A: V \rightarrow V$ and a cone $K \subset V$, the image $A(K)$ and the preimage $A^{-1}(K)$ are cones. The interior V^0 of V^+ is formed by those $v \in V^+$ such that $v_i > 0$ for every $1 \leq i \leq n$, we write $0 \ll v$ for $v \in V^0$.

Theorem.

Let p_1, \dots, p_n be deterministic *w-self-supportive* strategies and $G^{(w)} = (g^{(w)}(p_i, p_j))$ be the terminal pay-off matrix. The following are equivalent:

(a)

There exists a society $S(u_i, p_i | i = 1, \dots, n)$ which is *w-successful*.

(b)

There exists a vector $0 \ll u \in R^n$ such that $G^{(w)}u \gg 0$.

(c)

$(G^{(w)})^{-1}(V^+) \cap V^+$ is a solid cone.

Proof.

(a) \Rightarrow (b): If $S(u_i : p_i | i = 1, \dots, n)$ is w -successful, then $G^{(w)}u \geq g^{(w)} \gg 0$ because all p_i are self-supportive strategies. For the converse, observe that by continuity, we may assume that $0 \ll u \in Q^n$ and $G^{(w)}u \gg 0$. Then for some natural number m , we get $0 \ll v := mu \in N^n$ and $G^{(w)}v = mG^{(w)}u \geq g^{(w)}$.

(b) \Rightarrow (c): Clearly, $(G^{(w)})^{-1}(V^+) \cap V^+$ is a cone and there is a number $\varepsilon > 0$ such that for any vector $v \in R^n$, $\|v - u\| < \varepsilon$, then $v \gg 0$ and $G^{(w)}v \gg 0$. Then the ball $B_{\varepsilon}(u) \subset (G^{(w)})^{-1}(V^+) \cap V^+$ and the cone $(G^{(w)})^{-1}(V^+) \cap V^+$ is solid. The implication (c) \Rightarrow (b) is clear. \square

4.3.

Let p_1, \dots, p_n be deterministic strategies. Consider the symmetric matrix $A^{(w)}(p_1, \dots, p_n) := \frac{1}{2} (G^{(w)} + (G^{(w)})^t)$ and

$$q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n) = \sum_{i=1}^n g^{(w)}(p_i : p_i) X_i^2 + \sum_{i < j} (g^{(w)}(p_i : p_j) + g^{(w)}(p_j : p_i)) X_i X_j$$

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the associated quadratic form.

Recall from the Introduction that the society $S = S(u_i : p_i | i = 1, \dots, n)$ is w -macro-successful if $\sum_{x \in S} g_S^{(w)}(x) = \sum_{i=1}^n u_i g_S^{(w)}(x_i) \geq 0$, for any selection of individuals x_i using the strategy p_i ($1 \leq i \leq n$). Obviously a w -successful society is w -macro-successful.

Corollary.

Let p_1, \dots, p_n be strategies and let $S = S(u_i : p_i | i = 1, \dots, n)$ be a society. Then S is w -macro-successful if and only if $q_{(p_1, \dots, p_n)}^{(w)}(u_1, \dots, u_n) \geq u^t g^{(w)}$. Moreover, this number is positive if all strategies p_1, \dots, p_n are w -self-supportive.

Proof.

Let x_i be an individual in S using strategy p_i . Observe that

$$\begin{aligned} \sum_{x \in S} g_S^{(w)}(x) &= \sum_{i=1}^n u_i g_S^{(w)}(x_i) \\ &= \sum_{i=1}^n u_i (u_i - 1) g^{(w)}(p_i : p_i) + \sum_{i < j} u_i u_j [g^{(w)}(p_i : p_j) + g^{(w)}(p_j : p_i)] \\ &= q_{(p_1, \dots, p_n)}^{(w)}(u_1, \dots, u_n) - u^t g^{(w)}. \end{aligned}$$

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The claim follows. \square

4.4.

We shall say that the deterministic strategies p_1, \dots, p_n are w -compatible if for vectors $(u_1, \dots, u_n) \in N^n$ with big enough $u = \sum_{i=1}^n u_i$ we get societies $S(u_i : p_i | i = 1, \dots, n)$ which are w -macro-successful. We characterize compatible strategies by properties of the associated quadratic form $q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$ and then by simple properties of the symmetric matrix $A^{(w)}(p_1, \dots, p_n)$.

Theorem.

Let p_1, \dots, p_n be deterministic strategies. Then the following are equivalent:

(a)

p_1, \dots, p_n are w -compatible.

(b)

$q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$ is weakly positive, i.e. for every vector $0 \neq v \in R^n$ with non-negative coordinates we have $q_{(p_1, \dots, p_n)}^{(w)}(v) > 0$.

Proof.

(a) \Rightarrow (b): Assume p_1, \dots, p_n are w -compatible and let $0 \neq v \in N^n$. Consider $m \in N$ such that $w = mv$ has $\sum_{i=1}^n w_i$ big enough. Then $S(w_i : p_i | i = 1, \dots, n)$ is w -macro-successful and by 4.3,

$$0 < q_{(p_1, \dots, p_n)}^{(w)}(w) = m^2 q_{(p_1, \dots, p_n)}^{(w)}(v) \quad \text{and} \quad 0 < q_{(p_1, \dots, p_n)}^{(w)}(v).$$

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(b) \Rightarrow (a): Consider the compact set $C = \{v \in R^n : 0 \leq v \text{ and } \|v\| = 1\}$. The hypothesis implies that

the form $q_{(p_1, \dots, p_n)}^{(w)}(X_1, \dots, X_n)$ reaches a minimum $\gamma > 0$ in C and the linear form $\sum_{i=1}^n g^{(w)}(p_i : p_i) X_i$ reaches a maximum δ . Then for any vector $u = (u_1, \dots, u_n)^t \in N^n$ with

$$q_{(p_1, \dots, p_n)}^{(w)}(u) < u^t g^{(w)},$$

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we have

$$\gamma \|u\|^2 \leq q_{(p_1, \dots, p_n)}^{(w)}\left(\frac{u}{\|u\|}\right) \|u\|^2 = q_{(p_1, \dots, p_n)}^{(w)}(u) < u^t g^{(w)} = \|u\| \left(\frac{u^t}{\|u\|}\right) g^{(w)} \leq \|u\| \delta.$$

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Therefore $\|u\| \leq \delta/\gamma$ and only finitely many vectors $u \in N^n$ may have this property. Therefore

for $\sum_{i=1}^n u_i \gg 0$ we have $q_{(p_1, \dots, p_n)}^{(w)}(u) \geq u^t g^{(w)}$, that is, $S(u_i : p_i | i = 1, \dots, n)$ is w -macro-successful. \square

4.5.

There are good criteria to decide whether or not the quadratic form $q(x) = x^t A x$, associated to a symmetric $n \times n$ matrix A , is weakly positive. The following is a simple generalization of a result by Zel'dich [10] (see also [3]).

Proposition.

Let A be a symmetric matrix and $q(x) = x^t A x$ the associated quadratic form. The following are equivalent:

(a)

$q(x)$ is weakly positive.

(b)

For every principal submatrix $B = A \begin{pmatrix} i_1 & \dots & i_r \\ i_1 & \dots & i_r \end{pmatrix}$ of A , either $\det B > 0$ or the adjoint matrix $\text{ad}(B)$ is not positive (that is, it has an entry ≤ 0).

Proof.

(a) \Rightarrow (b): Let B be a principal submatrix of A . Suppose that $\text{ad}(B)$ is positive. By Perron theorem, $\text{ad}(B)v = \rho v$ for a vector $0 \neq v \geq 0$ and the spectral radius $\rho > 0$. Then $0 < q(v) = v^t B v = \rho^{-1} v^t B \text{ad}(B)v = \rho^{-1} \det B v^t v$ and $\det B > 0$.

(b) \Rightarrow (a): We show that $q(x)$ is weakly positive by induction on n . Since property (b) is inherited to principal submatrices, we get that the restriction $q^{(i)}$ associated to the principal submatrix $A^{(i,i)}$ is weakly positive, $i = 1, \dots, n$.

Assume that q is not weakly positive. Then there is a vector $0 \ll w$ with $q(w) \leq 0$.

We claim that $q^{(i)}$ is positive for all $1 \leq i \leq n$. Otherwise, $q^{(i)}(x) \leq 0$ for some vector $0 \neq x \in R^{n-1}$. Since $q^{(i)}$ is weakly positive, then $x_a > 0$ and $x_b < 0$ for indices a, b . We may consider $y \in R^n$ with $y_i = x_j$ for $j \neq i$ and $y_i = 0$.

We find two points $w + \lambda_1 y$ and $w + \lambda_2 y$ in the boundary ∂V^+ of the positive cone V^+ in R^n . Hence the parabola $q(w + \lambda y) = q(w) + \lambda w^t A y + \lambda^2 q(y)$ takes values > 0 (resp. $\leq 0, > 0$) in $\lambda = \lambda_1$ (resp. $\lambda = 0, \lambda = \lambda_2$). Hence $q(w + \lambda y)$ takes positive values for $\lambda \geq \lambda_2$. Therefore $0 < q(y) = q^{(i)}(x) \leq 0$, a contradiction proving the claim.

In particular, every proper principal submatrix B of A has $\det B > 0$. Since A is not positive, $\det A \leq 0$. By hypothesis, $\text{ad}(B)$ is not positive. Assume that the j th row v of $\text{ad}(A)$ is not positive. Choose $\lambda \geq 0$ such that $0 \leq \lambda w + v$ lies on ∂V^+ . Hence

$$0 < q(\lambda w + v) = \lambda^2 q(w) + \lambda w^t A v + q(v) \leq \lambda (\det A)_{w,j} + (\det A)_{v,j} \leq (\det A) (\det A^{(i,i)}) \leq 0,$$

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since by the claim $q^{(i)}$ is positive. This contradiction completes the proof of the result. \square

5. Dependence on the admissible weights

5.1.

In this section we shall discuss in which way the former results depend on the fixed admissible weight w with which the IPD is played. The discussion is motivated by remarks of a referee of the paper.

Let $w = (T, R, P, S)$ satisfying be an *admissible weight*. Given p and p' two deterministic strategies, we denote by $g^{(w)}(p:p')$ the relative pay-off of p playing the IPD with initial conditions w against p' .

Lemma.

The set of admissible parameters $w \in R^4$ satisfying $g^{(w)}(p:p') \geq 0$ together with the origin 0 form a cone $C(p:p')$ in R^4 . Moreover:

- (i) The cone $C(p:p')$ is either 0 or a solid cone.
- (ii) If $C(p:p') = 0$ then for any admissible tuple w we get $g^{(w)}(p:p') \leq g^{(w)}(p':p)$, that is, the pay-off of an individual playing the strategy p is lower than the pay-off of another playing the strategy p' , independently of the initial conditions.
- (iii) If $C(p:p')$ is a solid cone then the point $u = (1, 1, 0, -1)$ belongs to the topological closure of $C(p:p')$.

Proof.

Observe that for w, w' admissible parameters and $r > 0$ we get admissible parameters $w + w'$ and rw such that $g^{(w+w')}(p:p') = g^{(w)}(p:p') + g^{(w')}(p:p')$ and $g^{(rw)}(p:p') = rg^{(w)}(p:p')$. Hence $C(p:p')$ is a cone.

Let a (resp. b, c and d) be the number of arrows in the tournament-cycle of $t(p:p')$ with outcome $1:1$ (resp. $0:0, 0:1$ and $1:0$). Then for $w = (T, R, P, S)$ we

$$\text{have } \frac{g^{(w)}(p:p')}{a+b+c+d} = \frac{aR+bP+cT+dS}{a+b+c+d}.$$

(i) Assume that $C(p:p')$ is not trivial and let $0 \neq w = (T, R, P, S) \in C(p:p')$. Then $T > R > 0 > P > S$. If both $a = 0$ and $c = 0$, then also $b = 0 = d$. Therefore $C(p:p') = R^4$. Assume that $a > 0$ then, by slightly modifying R , we get a point $w' \in C(p:p')$ such that $g^{(w')}(p:p') > 0$. We do not lose generality assuming that $w = w'$. Then there are small values $r > 0$ such that any $E \in R^4$ with norm $|E| < r$ satisfies $a(R + \epsilon_1) + b(P + \epsilon_2) + c(T + \epsilon_3) + d(S + \epsilon_4) > 0$, that is the sphere with center w and radius r lies in $C(p:p')$. Therefore $C(p:p')$ is a solid cone.

(ii) With the notation above, $C(p:p') = 0$ implies $a = 0 = c$. Then for any admissible w we get $g^{(w)}(p':p) = \frac{bP+dT}{b+d} \geq \frac{bP+dS}{b+d} = g^{(w)}(p:p')$.

(iii) Let $0 \neq w = (T, R, P, S) \in C(p:p')$. As in (i) we may assume that $g^{(w)}(p:p') > 0$. Then $0 < aR + bP + cT + dS \leq (a+c)T + dS \leq [-(a+c) + d]S$, which implies that $a+c > d$. There is a sequence of admissible tuples $w_n = (1 + \frac{1}{n}, 1 + \frac{1}{n}, \frac{1}{n}, -1 + \frac{1}{n})$, for n big enough, with $g^{(w_n)}(p:p') > 0$. \square

5.2.

Let $p = (p_1, \dots, p_n)$ be a sequence of strategies. We introduce an equivalence relation \sim_p in the set of all admissible weights in the following way: for $w = (T, R, P, S)$ and $w' = (T', R', P', S')$ admissible weights we write $w \sim_p w'$ if for any couple $1 \leq i, j \leq n$ the inequality $g^{(w)}(p_i, p_j) > 0$ (resp. $= 0, < 0$) happens exactly when $g^{(w')}(p_i, p_j) > 0$ (resp. $= 0, < 0$) holds. Observe that this means that w and w' belong to the same sequence of half-spaces in R^4 determined by the hyperplanes $H_{i,j}$ defined by the linear equation $a_{i,j}R + b_{i,j}P + c_{i,j}T + d_{i,j}S = 0$, where $a_{i,j}$ (resp. $b_{i,j}, c_{i,j}, d_{i,j}$) denotes the number of the tournament-cycle of $t(p_i, p_j)$ with outcome $1:1$ (resp. $0:0, 0:1, 1:0$).

Proposition.

There is only a finite number of \sim_p -equivalence classes of admissible weights. For each equivalence class C the topological closure \overline{C} in R^4 is a convex cone. The cone \overline{C} is solid if and only if C is an open set.

Proof.

The complement in R^4 of the union of all hyperplanes $H_{i,j}$, for pairs $1 \leq i, j \leq n$, is formed by a finite number of open subsets U_1, \dots, U_s . For any $i = 1, \dots, s$, two points in the open set U_i are \sim_p -equivalent. The other equivalence classes are the different walls of the topological closures of the U_i , for $i = 1, \dots, s$.

Clearly, if C is an equivalence class, then its closure satisfies: (i) $0 \in \overline{C}$; (ii) for $v \in \overline{C}$ and $\lambda \geq 0$, then $\lambda v \in \overline{C}$ and (iii) if $v, v' \in \overline{C}$, then $v + v' \in \overline{C}$. If C is open, it clearly contains a basis of R^4 and \overline{C} is solid. For the converse, observe that if \overline{C} is solid, then there is an open ball $B_r(x)$ contained in C . Then $C = U_i$ for some $1 \leq i \leq s$. \square

Any \sim_p -equivalence class whose topological closure contains the (non-admissible) weight $(1, 1, -1, -1)$ is called a *canonical class*.

Corollary.

There is a canonical class which is open. If w is an admissible weight in a canonical class, then for any pair $1 \leq i, j \leq n$, the inequality $g^{(w)}(p_i, p_j) \geq 0$ implies that $a_{i,j} + b_{i,j} \geq c_{i,j} + d_{i,j}$ in the tournament-cycle of $t(p_i, p_j)$.

Proof.

Observe that the points $(1 + \lambda_1, 1 - \lambda_2, -1 + \lambda_3, -1 - \lambda_4)$ with $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 1$ form a set of admissible weights that cannot be contained in a finite set of non-solid cones. Hence some of these weights lie in a canonical class. The second claim follows by continuity. \square

5.3.

Let again $p = (p_1, \dots, p_n)$ be a sequence of strategies and consider a society $S = S(\{u_i: p_i \mid i = 1, \dots, n\})$. Let x_i be an individual in S playing the strategy p_i , for $i = 1, \dots, n$. As above, define $\sim_{(S,i)}$ be the equivalence relation in the set of admissible weights such that $w \sim_{(S,i)} w'$ if both x_i is w -successful and w' -successful in the society S . By the arguments in [5.2](#),

there are finitely many $_{-(S,i)}$ -equivalence classes $c_{i,1}, \dots, c_{i,s_i}$ of admissible weights. Consider a set C of admissible weights of the form $\bigcap_{i=1}^n C_{i,t_i}$, for some $1 \leq t_1 \leq s_1, \dots, 1 \leq t_n \leq s_n$, then two weights w, w' in C satisfy the following properties:

- (a) Let x be an individual in S , then x is w -successful in S if and only if it is w' -successful in S . Denote by $_{ES}(w)$ the set of indices i such that an individual x playing the strategy p_i is w -successful. Hence $_{ES}(w) = _{ES}(w')$.
- (b) The society S is w -successful if and only if it is w' -successful. In that case $_{ES}(w) = \{1, \dots, n\}$.
- (c) Moreover, the topological closure \overline{C} of C in R^4 is a cone.

Corollary.

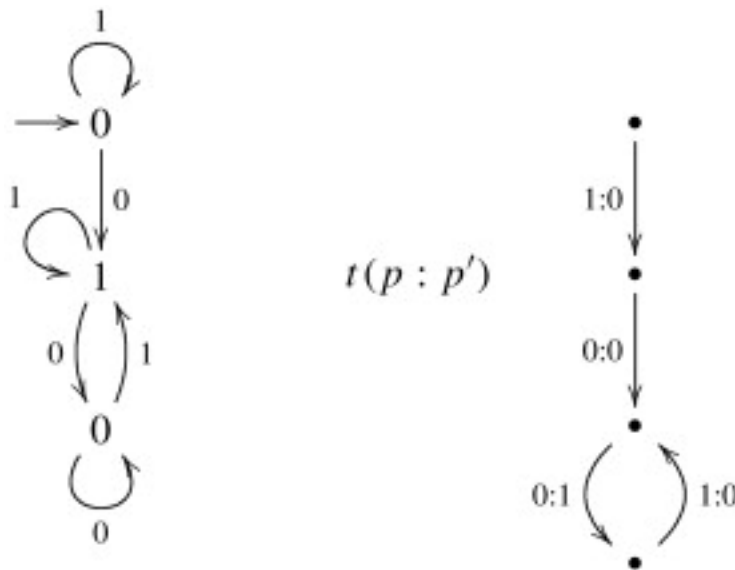
There is a finite partition c_1, \dots, c_m of the admissible weights such that the following holds:

- (i) the closure $\overline{c_i}$ of each c_i is a convex cone;
- (ii) for any two admissible weights w, w' , $_{ES}(w) = _{ES}(w')$ if and only if w and w' belong to the same set c_i for some i .

6. Examples

6.1.

For the next examples we fix values $T=2, R=1, P=-1, S=-3$. Consider p the strategy tit-for-tat and p' the strategy given by the digraph



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The tournament $t(p : p')$ is indicated above. The pay-off matrix G is

$$G = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}.$$

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Then $G\begin{pmatrix} x \\ y \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is satisfied when

$$x \geq \frac{1}{2}y + 1 \geq \frac{1}{2} \left[\frac{1}{2}x + 1 \right] + 1 = \frac{1}{4}x + \frac{3}{2}.$$

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For example, a society $S(x:p, y:p')$ with $x=y \geq 2$ is successful, while $S(x:p, 2x:p')$ is not successful.

The associated quadratic form is

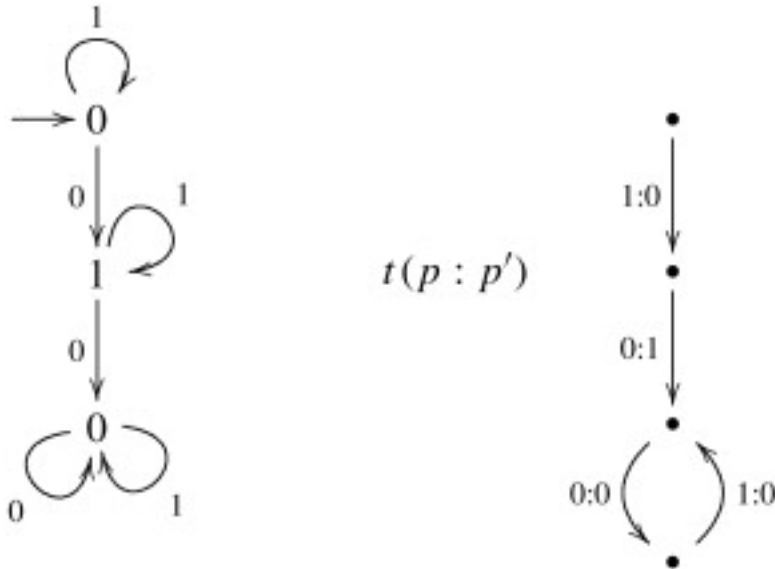
$$q(x, y) = x^2 - xy + y^2 = \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2$$

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which is positive. Then a society $S(x:p, y:p')$ with $x+y \gg 0$ is macro-successful (that is, p and p' are compatible).

6.2.

Consider p the strategy PAV and p' the strategy given by the digraph



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With the values of T, R, P and S as in [6.1](#), the pay-off matrix G is

$$G = \begin{bmatrix} 1 & -\frac{5}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

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Then $G^{-1}V^+ \cap V^+ = \{(x, y) : x \geq \frac{5}{2}y \geq 0\}$. For example, $S(3y : p, y : p')$, with $y \geq 2$, is a successful society.

The associated quadratic form is

$$q(x, y) = x^2 - 2xy + y^2 = (x - y)^2$$

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which is not weakly positive. Therefore p and p' are not compatible.

6.3.

Consider p the intolerant strategy $_{10}$ and $p' = (\{b_0, b_1, \dots, b_m\}, f_0, f_1, S)$ any strategy. Observe that in case $_{\alpha_i} = 1 : 0$ in the tournament $t(p : p')$, then $_{\alpha_j} = 0 : _{\alpha_j}$ for some $_{\alpha_j} \in \{0, 1\}$ and any $j \geq i + 1$.

Then $g(p : p') = \frac{bP+cT}{b+c}$ and $g(p' : p) = \frac{bP+cS}{b+c}$ for some $b, c \geq 0$ and $b + c = c(p : p')$. With the assignment of parameters given

in 6.1, $g(p : p') = \frac{-b+2c}{b+c}$ and $g(p' : p) = \frac{-b-3c}{b+c}$. Otherwise, all $_{\alpha_i} = 1 : 1$ ($i \geq 1$) and $g(p : p') = g(p' : p) = R = 1$. In the first case, the associated

quadratic form is

$$q(x, y) = x^2 - \frac{2b+c}{b+c}xy + g(p' : p)y^2$$

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which is weakly positive if and only if $g(p' : p') = 1$ and $b = 0$. In conclusion, given a self-supportive strategy p' , the strategies p and p' are compatible if either $g(p : p') = 1 = g(p' : p)$ and then any society $S(x : p, y : p')$ is successful or if $g(p : p') = 2$, $g(p' : p) = -3$ and $g(p' : p') = 1$ and then, only societies $S(x : p, y : p')$ with $y \geq 3x + 1$ are successful.

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